

# Complexity of a Single Face in an Arrangement of $s$ -Intersecting Curves\*

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## Abstract

Consider a face  $F$  in an arrangement of  $n$  Jordan curves in the plane, no two of which intersect more than  $s$  times. We prove that the combinatorial complexity of  $F$  is  $O(\lambda_s(n))$ ,  $O(\lambda_{s+1}(n))$ , and  $O(\lambda_{s+2}(n))$ , when the curves are bi-infinite, semi-infinite, or bounded, respectively;  $\lambda_k(n)$  is the maximum length of a Davenport-Schinzel sequence of order  $k$  on an alphabet of  $n$  symbols.

Our bounds asymptotically match the known worst-case lower bounds. Our proof settles the still apparently open case of semi-infinite curves. Moreover, it treats the three cases in a fairly uniform fashion.

## 1 Introduction

In this paper we study the maximum complexity of a single face in an arrangement of curves in the plane, no two of which intersect more than  $s$  times; see below. We will do this through an extensive use of Davenport-Schinzel sequences, which were first introduced by Davenport and Schinzel in 1965 [DS65]. They were motivated, curiously enough, by a problem in differential equations.

**Definition.** Let  $n, s$  be positive integers. A sequence  $U = \langle u_1, \dots, u_m \rangle$  over an alphabet of size  $n$  is a *Davenport-Schinzel sequence* of order  $s$  on an alphabet of  $n$  symbols, or *DS( $n, s$ )-sequence*, for short, if it satisfies the following conditions:

1.  $u_i \neq u_{i+1}$ , for each  $1 \leq i < m$ .
2. There do not exist  $s + 2$  indices  $1 \leq i_1 < i_2 < \dots < i_{s+2} \leq m$  such that  $u_{i_1} = u_{i_3} = u_{i_5} = \dots = a$ ,  $u_{i_2} = u_{i_4} = u_{i_6} = \dots = b$ , for some distinct symbols  $a$  and  $b$ .

We denote by  $\lambda_s(n)$  the length of the longest *DS( $n, s$ )-sequence*.

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Davenport and Schinzel were able to establish a connection between these sequences and lower envelopes of collections of functions [DS65, SA96]. The next significant step in studying these sequences was taken by Szemerédi in 1974, who established improved upper bounds on the length of Davenport-Schinzel sequences [Sze74]. In 1983, Atalah's work was the first step in establishing DS-sequences as a fundamental tool in computational and combinatorial geometry [Ata85]. The fundamental question that was still unanswered was determining the asymptotic growth rate of the functions  $\lambda_s(n)$ , for any fixed  $s$ . For  $s = 1$  and  $s = 2$ , this is very easy ( $\lambda_1(n) = n$  and  $\lambda_2(n) = 2n - 1$ ) but already for  $s = 3$ , this question is highly nontrivial. In 1986, Hart and Sharir showed that the maximum length of any  $DS(n, 3)$ -sequence is  $O(n\alpha(n))$ , where  $\alpha(n)$  is the very slowly growing inverse of Ackermann's function [HS86]. In 1989, Agarwal, Sharir, and Shor completed this classification by showing nearly tight, nearly linear bounds for all fixed  $s$  [ASS89]. Davenport-Schinzel sequences have proven to be very useful in providing tighter methods of analysis for many problems in discrete and computational geometry [SA96, AS00].

In this paper, we are interested in the following three types of curves. An *unbounded Jordan curve* is the image of an open unit interval under a topological embedding into  $\mathbb{R}^2$ , such that it separates the plane. A *semi-infinite Jordan curve* is the image of a half-open unit interval under a topological embedding into  $\mathbb{R}^2$ , such that the image is unbounded with respect to the standard Euclidean norm. A *bounded Jordan curve* (or *Jordan arc*) is the image of a closed unit interval under a topological embedding into  $\mathbb{R}^2$ .

Let  $\Gamma_0$ ,  $\Gamma_1$ , and  $\Gamma_2$  be collections of  $n$  bi-infinite, semi-infinite, and bounded Jordan curves in the plane, respectively, such that any two curves in  $\Gamma_i$  intersect at most  $s$  times, for some fixed constant  $s > 0$ . (The subscript of  $\Gamma_i$  signifies the number of finite endpoints of each curve in this collection.)

**Definition.** [Ede87, SA96] The *arrangement*  $A(\Gamma_i)$  of  $\Gamma_i$  is the planar subdivision induced by the arcs of  $\Gamma_i$ . Thus  $A(\Gamma_i)$  is a planar map whose *vertices* are the endpoints of curves of  $\Gamma_i$ , if any, and their pairwise intersection points. The *edges* are maximal connected portions of the curves that do not contain a vertex. The *faces* are the connected components of  $\mathbb{R}^2 - \bigcup \Gamma_i$ .

The *combinatorial complexity* of a face  $F$  of  $A(\Gamma_i)$  is the total number of vertices and edges of  $A(\Gamma_i)$  along its boundary  $\partial F$ . A feature on  $\partial F$  is counted in the complexity as many times as it appears.

We are interested in studying the maximum combinatorial complexity of a single face of  $A(\Gamma_i)$ . Schwartz and Sharir showed that the combinatorial complexity of a single face of  $A(\Gamma_0)$  is at most  $\lambda_s(n)$  [SS90]. Sharir *et al.* showed that the combinatorial complexity of a single face of  $A(\Gamma_2)$  is  $O(\lambda_{s+2}(n))$  [GSS88]. (See Nivasch [Niv09] for some very recent progress in this subject.) However, the two proofs provided are very different, which is somewhat unsatisfying.

It has also been conjectured that the combinatorial complexity of a single face of  $A(\Gamma_1)$  is  $O(\lambda_{s+1}(n))$ . There has been some work to suggest that this is true. For instance, Alevizos, Boissonnat, and Preparata showed that the complexity of a single face in an arrangement of rays is linear [PA90]; this is the case when  $s = 1$ . In this paper, we prove the following theorem:

**Theorem 1.** *The combinatorial complexity of a face  $F$  in an arrangement of  $n$  bi-infinite, semi-infinite, or bounded Jordan curves, no two of which intersect more than  $s$  times, is  $O(\lambda_s(n))$ ,  $O(\lambda_{s+1}(n))$ , and  $O(\lambda_{s+2}(n))$ , respectively.*

These upper bounds are tight in the worst case. This easily follows from the fact that the complexity of the lower envelope of the collection of functions defined by curves of  $\Gamma_i$  (these functions are

partially defined for  $i = 1$  and for  $i = 2$ ), in the special case where the curves are  $x$ -monotone, is a lower bound on the combinatorial complexity of a single face of  $A(\Gamma_i)$  [Ata85, SA96]; the maximum complexity of such an envelope is  $\Theta(\lambda_s(n))$ ,  $\Theta(\lambda_{s+1}(n))$ , and  $\Theta(\lambda_{s+2}(n))$ , respectively.

The result is known for bi-infinite curves and for Jordan arcs [GSS88, SS90], but not, to the best of our knowledge, for semi-infinite curves. The advantage of our proof is that firstly it settles the previously mentioned conjecture and secondly it treats all three cases in a reasonably uniform manner.

Our paper is organized as follows. In Section 2 we prove a purely combinatorial auxiliary fact (Fact 2). Section 3 contains some preliminary modifications to the geometric problem. In Section 4 we prove bounds on the maximum complexity of an unbounded face in  $A(\Gamma_i)$ . Finally, in Section 5 we transform any bounded face of  $A(\Gamma_i)$  into an unbounded one without an asymptotic increase in its complexity. This implies bounds on the maximum complexity of a bounded face in  $A(\Gamma_i)$  and yields our main theorem.

## 2 A Combinatorial Fact

In this section, we state and prove a simple combinatorial fact about Davenport-Schinzel sequences. It or a close relative have been “in the folklore” of this area of research [SA96], although we have not been able to pin down a source where it was explicitly stated in this form. For completeness, we present a proof.

**Definition.** Given a sequence  $S$  over an alphabet  $\Sigma$ , for  $\Lambda \subseteq \Sigma$ ,  $S|_{\Lambda}$  denotes the sequence obtained by deleting from  $S$  all symbols not in  $\Lambda$ .

**Definition.** Let  $\Sigma$  be an alphabet. Denote by  $\Sigma^*$  the set of all finite sequences over  $\Sigma$ . We define an operation  $\diamond : \Sigma^* \rightarrow \Sigma^*$  as follows. Let  $X \in \Sigma^*$ .  $X^\diamond$  is obtained from  $X$  by simply collapsing each subsequence of consecutive identical elements to a single element, e.g.,  $\langle \dots, a, b, b, b, b, c, c, d, \dots \rangle$  would be collapsed to  $\langle \dots, a, b, c, d, \dots \rangle$ .

**Definition.** Let  $\Sigma_1$  and  $\Sigma_2$  be disjoint alphabets, let  $k \geq 1$  be an integer, and let  $X = \langle x_1, \dots, x_m \rangle$  be a sequence over  $\Sigma_1 \cup \Sigma_2$ . We say that  $X$  is  $k$ -friendly under  $(\Sigma_1, \Sigma_2)$  if the following condition holds:

- (\*) There do not exist  $k + 1$  consecutive indices  $1 \leq i, i + 1, \dots, i + k \leq m$  such that  $x_i = x_{i+2} = x_{i+4} = \dots = a$ ,  $x_{i+1} = x_{i+3} = x_{i+5} = \dots = b$ , with  $a \in \Sigma_1$  and  $b \in \Sigma_2$ , or vice versa.

**Fact 2.** *If a sequence  $X$  is  $k$ -friendly under  $(\Sigma_1, \Sigma_2)$ , no two consecutive symbols of  $X$  are the same, and  $(X|_{\Sigma_1})^\diamond$  and  $(X|_{\Sigma_2})^\diamond$  are both  $DS(n, s)$ -sequences, then  $|X| = O(k\lambda_s(n))$ .*

*Proof.* Let  $L' = X|_{\Sigma_1}$ ,  $L = (L')^\diamond$ ,  $R' = X|_{\Sigma_2}$ , and  $R = (R')^\diamond$ . It is clear that  $|X| = |L| + |R| + (|L'| - |L|) + (|R'| - |R|)$ . Since  $|L|, |R| \leq \lambda_s(n)$ , without loss of generality, it is sufficient to bound  $\Delta_L = |L'| - |L|$ .  $\Delta_L$  is the number of elements that were deleted from  $L'$  by the  $\diamond$  operation. Suppose that a subsequence  $\langle a, b, \dots, b, c \rangle$  of  $|L'|$  was collapsed to  $\langle a, b, c \rangle$  in  $L$  (collapses at the beginning and at the end of  $L$  are handled similarly). Now the only way that this could have happened is that in  $X$ , between every two corresponding consecutive elements  $b$ , there was a sequence of one or more elements all from  $R'$ ; denote such a sequence by  $\xi_i$ . Let  $T = \langle b, \xi_1, b, \xi_2, \dots, b \rangle$ . We charge each element  $b$  in  $T$  to an element of  $\xi_i$  following it, such that if possible it is different from the element

of  $\xi_{i-1}$  that was charged for the previous occurrence of  $b$ . If it were always possible to do so, then all the elements of  $R'$  that have been charged would be preserved when  $R'$  were transformed into  $R$  and each one would have been charged only once, so we could bound  $\Delta_L$  by  $|R|$ . The only time that it is not possible is when there is a subsequence of  $X$  of the form  $\langle \dots, b, r, b, r, \dots \rangle$ , where  $r$  is an element of  $R$ . Since  $X$  is *k-friendly under*  $(\Sigma_1, \Sigma_2)$ , the length of such a subsequence is no larger than  $k$ . It now easily follows that in the above charging scheme, an element of  $R$  may be charged up to  $O(k)$  times, so  $\Delta_L = O(k\lambda_s(n))$ . Therefore  $|X| = |L| + |R| + \Delta_L + \Delta_R = O(k\lambda_s(n))$ .  $\square$

### 3 Geometric Preliminaries

We now return to the geometric problem. Recall that we start with a set  $\Gamma_i$  of curves in the plane, no two intersecting pairwise more than  $s$  times. In order to state our argument, no modifications will be required for curves in  $\Gamma_0$ , since only one side of any curve in  $\Gamma_0$  can appear on the boundary of  $F$ . However some modifications will be needed for the curves in  $\Gamma_1$  and  $\Gamma_2$ , which we describe below.

Let  $a = a(\gamma)$  be the endpoint of a curve  $\gamma \in \Gamma_1$ . Let  $\gamma^+$  be the directed curve that constitutes the “right side” of  $\gamma$  oriented from  $a$  to infinity and let  $\gamma^-$  be the “left” side of  $\gamma$  oriented from infinity to  $a$ . Let  $a = a(\gamma)$  and  $b = b(\gamma)$  be the two endpoints of a curve  $\gamma \in \Gamma_2$  that are chosen arbitrarily and fixed. Let  $\gamma^+$  (the “right” side) be the directed curve  $\gamma$  oriented from  $a$  to  $b$  and let  $\gamma^-$  (the “left” side) be the directed curve  $\gamma$  oriented from  $b$  to  $a$ .

#### 3.1 Associate a sequence with a face

Let  $F_i$  of  $A(\Gamma_i)$  be an unbounded face and let  $C_i$  be a connected component of  $\partial F_i$ . In this subsection, we show how to associate a sequence of curves with  $C_i$ .

- For  $A(\Gamma_2)$ , we traverse  $C_2$ , keeping  $F_2$  on the right. Let  $S_2 = \langle s_1, s_2, \dots, s_t \rangle$  be the circular sequence of oriented curves in  $\Gamma_2$  in the order in which they appear along  $C_2$ . If during the traversal we meet the curve  $\gamma$  with endpoints  $a = a(\gamma)$  and  $b = b(\gamma)$ , and follow it from  $a$  to  $b$  (respectively  $b$  to  $a$ ), we add  $\gamma^+$  (respectively  $\gamma^-$ ) to  $S_2$ .
- Observe that for  $A(\Gamma_0)$ ,  $C_0$  is not closed—it divides the plane into two connected components. This means that  $C_0$  naturally corresponds to a linear sequence of un-oriented curves. Again, we traverse it keeping  $F$  on the right. Denote this sequence by  $S_0$ .  $S_1$  is constructed analogously, as a sequence of *oriented* curves.

We will often abuse the notation slightly. Given a sequence of curves  $S_i$ , we will often isolate an alternating subsequence, say  $A = \langle \xi_j, \gamma_j, \xi_{j+1}, \gamma_{j+1}, \dots \rangle$ , where all  $\xi_j$  represent the appearances of the same curve  $\xi$  and all  $\gamma_j$  represent the appearances of the same curve  $\gamma$ . We will often treat  $\xi_j$  and  $\gamma_j$  as aliases for the edges of  $A(\Gamma_i)$  that correspond to those entries in  $S_i$ .

#### 3.2 Preliminary modification for curves in $\Gamma_1$

Let  $\Sigma_L$  and  $\Sigma_R$  be the alphabets consisting of the left symbols and right symbols, respectively. For notational purposes, let  $S_1^L = (S_1|_{\Sigma_L})^\diamond$  and  $S_1^R = (S_1|_{\Sigma_R})^\diamond$ . In this section, we prove a key lemma, which is a variation of the Circular Consistency Lemma [GSS88, SA96] below, and state an important observation.

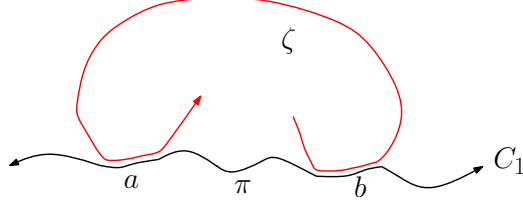


Figure 1: Proof of Linear Consistency Lemma.

**Lemma 3** (Linear Consistency Lemma).

- (a) *The portions of each arc  $\xi_i^+$  appear in  $S_1^R$  in the same order as their order along  $\xi_i^+$ ; analogous statement holds for  $S_1^L$ .*
- (b) *The portions of each arc  $\xi_i$  appear in  $S_0$  in the same or reverse order as compared to their order along  $\xi_i$ .*

*Proof.* We only argue (a), since (b) follows by an almost identical argument. Let  $a$  and  $b$  be portions of  $\xi^+$  that occur in  $S_1^R$  in that order. Assume that along  $\xi^+$ ,  $a$  follows  $b$ ; refer to Fig 1. Denote by  $\pi$  the portion of  $C_1$  connecting  $a$  to  $b$ . Denote by  $\zeta$  the portion of  $\xi^+$  from  $b$  to  $a$ . Now  $a\pi b\zeta$  is a closed contour. It is easy to verify that the infinite “end” of  $\xi^+$  must be enclosed in this closed contour, which is a contradiction.  $\square$

We now make the following simple but important observation. It can easily be checked that  $S_1$  is  $k$ -friendly under  $(\Sigma_L, \Sigma_R)$ , for some  $k = O(s)$ . (Indeed, the existence of a contiguous subsequence of the form  $\langle \zeta^-, \xi^+, \zeta^-, \xi^+, \dots \rangle$  of length  $s + 2$ , where  $\zeta^- \in \Sigma_L$  and  $\xi^+ \in \Sigma_R$ , would force  $s + 1$  distinct points of intersection of  $\zeta$  and  $\xi$ —a contradiction). This observation will be critical for the proof of Theorem 6.

### 3.3 Preliminary modification for curves in $\Gamma_2$

We will also need the following lemma.

**Lemma 4** (Circular Consistency Lemma [GSS88, SA96]). *The portions of each arc  $\xi_i^+$  (respectively  $\xi_i^-$ ) appear in  $S_2$  in a circular order consistent with their order along the oriented  $\xi_i^+$  (respectively  $\xi_i^-$ ). That is, there exists a starting point in  $S$ , which depends on  $\xi_i$ , such that if we read  $S$  in a circular order starting from that point, we encounter these portions in their order along  $\xi_i$ .*

We now perform a cutting of the circular sequence  $S_2$  as in [GSS88, SA96]. Consider  $S_2 = \langle s_1, \dots, s_t \rangle$  as a linear, rather than a circular sequence by breaking it at an arbitrary vertex. For each directed arc  $\gamma_i$ , consider the linear sequence  $V_i$  of all appearances of  $\gamma_i$  in  $S_2$ , arranged in the order they appear along  $\gamma_i$ . Let  $\mu_i$  and  $\nu_i$  denote, respectively, the index in  $S_2$  of the first and of the last element of  $V_i$ . For each arc  $\gamma_i$ , if  $\mu_i > \nu_i$ , we split the symbol  $\gamma_i$  into two distinct symbols  $\gamma_{i1}$  and  $\gamma_{i2}$ , and replace all appearances of  $\gamma_i$  in  $S_2$  between  $\mu_i$  and  $t$  (respectively, between 1 and  $\nu_i$ ) by  $\gamma_{i1}$  (respectively, by  $\gamma_{i2}$ ). Notice that by Lemma 4, we are able to split  $\gamma_i$  into two subarcs such that  $\gamma_{i1}$  represents the appearances of the first subarc and  $\gamma_{i2}$  represents the appearances of the second subarc. This splitting produces a sequence, of the same length as  $S_2$  on the alphabet of at most  $4n$  symbols. To simplify the notation, hereafter we refer to this new linear sequence as  $S_2$ .

To summarize:

- We did not modify  $S_0$ . It is a linear sequence of curves.
- $S_1$  is a linear sequence of oriented curves, from which we have derived two subsequences,  $S_1^L$  and  $S_1^R$ .
- After the cutting procedure,  $S_2$  is a linear sequence of oriented curves.

To arrive at our first geometric theorem, we need the following lemma [GSS88, SA96].

**Lemma 5** (Quadruple Lemma [GSS88, SA96]). *Consider a quadruple of consecutive elements in a fixed alternating subsequence of  $S_2$ . Let this quadruple be  $\langle \xi_1, \gamma_1, \xi_2, \gamma_2 \rangle$ , such that  $\xi_i$  and  $\gamma_j$  constitute portions of curves  $a$  and  $b$ , respectively. Let  $\pi_a$  be the portion of  $a$  connecting  $\xi_1$  to  $\xi_2$  and let  $\pi_b$  be the portion of  $b$  connecting  $\gamma_1$  to  $\gamma_2$ . Then  $\pi_a$  and  $\pi_b$  must intersect. Furthermore, this point of intersection is distinct for each such quadruple in this subsequence.*

Although stated for  $S_2$ , the Quadruple Lemma also holds for  $S_0$ ,  $S_1^R$ , and  $S_1^L$ .

## 4 Complexity of an Unbounded Face

**Theorem 6.** *The complexity of an unbounded face  $F$  in an arrangement of  $n$  (0) bi-infinite, (1) semi-infinite, or (2) bounded Jordan curves, no pair of which crosses more than  $s$  times, is  $O(\lambda_s(n))$ ,  $O(\lambda_{s+1}(n))$ ,  $O(\lambda_{s+2}(n))$ , respectively.*

*Proof.* Since  $\lambda_s(n)$  is at least linear, and no curve can appear on several connected components of  $\partial F$ , we can consider each component  $C$  separately and assume that all of the curves appear on it. Now consider the sequences  $S_0$  (case 0),  $S_1^R$  (case 1), and  $S_2$  (case 2). We claim that these are  $DS(n, s)$ -,  $DS(2n, s + 1)$ -, and  $DS(4n, s + 2)$ -sequences, respectively.

Let  $S$  be the sequence in question. We aim to argue that it is a  $DS$ -sequence. By construction,  $S$  does not contain any consecutive identical elements. Assume that it has an alternating subsequence of length  $l$ . Let this subsequence be  $A = \langle \xi_1, \gamma_1, \xi_2, \gamma_2, \dots \rangle$ , such that  $\xi_i$  and  $\gamma_j$  constitute portions of curves  $a$  and  $b$ , respectively. We argue that  $l$  cannot be too large.

By Lemma 5, consecutive quadruples of  $A$  force  $l - 3$  distinct crossings between  $a$  and  $b$ .

- In case 2, setting  $l = s + 4$  forces  $l - 3 = s + 1$  distinct intersections between  $a$  and  $b$ . This is a contradiction. Thus no such subsequence exists, and  $S_2$  is a  $DS(4n, s + 2)$ -sequence, implying that the complexity of  $F$  is  $O(\lambda_{s+2}(n))$ .
- In case 1, setting  $l = s + 3$  forces  $l - 3 = s$  distinct intersections between  $a$  and  $b$ . Now, consider the last quadruple in  $A$ . Without loss of generality, let it be  $\langle \xi_i, \gamma_i, \xi_{i+1}, \gamma_{i+1} \rangle$ . Let  $\pi$  be the portion of  $b$  connecting  $\gamma_i$  to  $\gamma_{i+1}$ . We claim that there must be an additional intersection between  $a$  and  $b$  at a point on  $\pi$  that we have not accounted for, so far. Let  $\theta \supset \xi_{i+1}$  be the portion of  $C$  connecting  $\gamma_i$  to  $\gamma_{i+1}$ ;  $\pi \cup \theta$  is a closed contour. Refer to Figure 2. Now traverse  $a$  in the infinite direction starting from  $\xi_{i+1}$ . Since  $a$  cannot cross  $\theta$ , during the traversal,  $a$  must intersect  $b$  at a point on  $\pi$ . By the Linear Consistency Lemma, however this intersection has not been accounted for and thus there are at least  $s + 1$  distinct intersections between  $a$

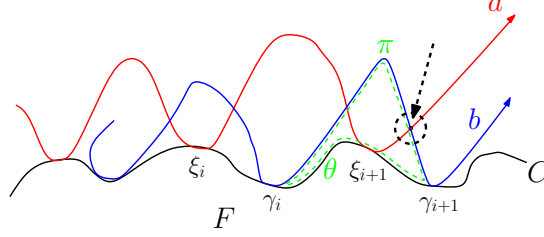


Figure 2: Semi-infinite curves.

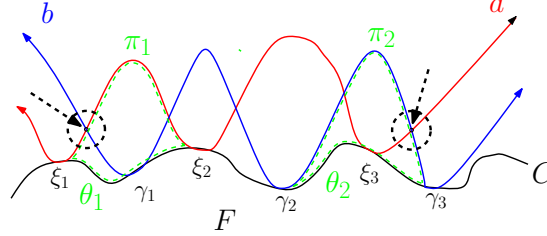


Figure 3: Bi-infinite curves.

and  $b$ . This, of course, is a contradiction and therefore  $S_1^R$  is a  $DS(2n, s+1)$ -sequence. By Theorem 2 and the discussion of Section 3.2,  $|S_2| = O(\lambda_{s+1}(n))$ , implying that the complexity of  $F$  is  $O(\lambda_{s+1}(n))$ .

- In case 0, setting  $l = s+2$  forces  $(s+2) - 3 = s-1$  distinct intersections between  $a$  and  $b$ . Now, consider the first and last quadruples of  $A$ . By the same argument as in case 1, there are now two additional distinct intersections between  $a$  and  $b$ , for a total of at least  $s+1$  distinct intersections—a contradiction. Thus  $S_0$  is a  $DS(n, s)$ -sequence and the combinatorial complexity of  $F$  is  $\lambda_s(n)$ . Refer to Figure 3.  $\square$

## 5 Complexity of an Arbitrary Face

The next step is to generalize our results to bounded faces. We will transform the problem while only increasing the complexity of a face  $F$ , to make  $F$  unbounded, and then apply our results from the previous section. More precisely, we will build a tunnel from  $F$  to the “outside”, so that after the transformation, the new face will be part of an unbounded face of the arrangement.

*Proof of Theorem 1.* If  $F$  is unbounded, we are done by Theorem 6. Consider a bounded face  $F$  in an arrangement of (a) bi-infinite, (b) semi-infinite, or (c) bounded Jordan curves. We assume that all the curves appear on  $\partial F$ , otherwise one or more curves can be deleted without affecting the complexity of  $F$ . Furthermore since  $\lambda_s(n)$  is at least linear, it is sufficient to argue the complexity of just one connected component of  $\partial F$ . Thus without loss of generality, we can assume that  $\partial F$  is connected.

### Step 1: Finding the site for the tunnel

- In cases (a) and (b), pick an arbitrary infinite edge of the arrangement, say of curve  $a$  and follow it until it first meets  $\partial F$ , say at point  $p$ , where it meets curve  $b$ . Denote

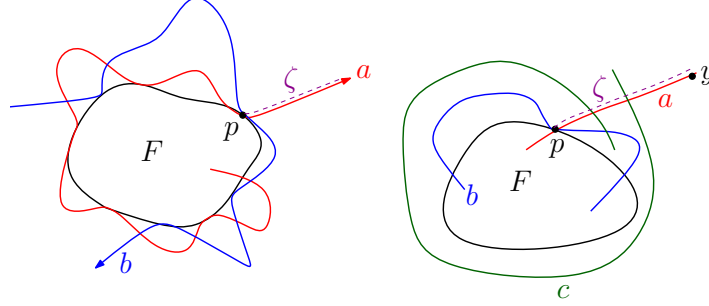


Figure 4: Finding the initial cut.  $\Gamma_1$  on the left and  $\Gamma_2$  on the right.

this portion of  $a$ , from infinity to  $p$ , by  $\zeta$ ;  $\zeta$  is the future site for our tunnel. Refer to Figure 4(left).

- In case (c), if an endpoint of a Jordan arc lies on the boundary of the unbounded cell, we start at this point. Otherwise, we pick an arbitrary edge of the infinite cell of the arrangement, cut the curve containing this edge into two curves, and move them slightly apart; this increases the number of curves by one. In both cases, we now have an endpoint  $y$  of a curve  $a$ . Now follow  $a$  from  $y$  to its first point of intersection  $p$  with  $\partial F$ , where it meets curve  $b$ . Denote this portion of  $a$ , from  $y$  to  $p$ , by  $\zeta$ ;  $\zeta$  is the future site for our tunnel. Refer to Figure 4(right).

**Step 2: Digging the tunnel** We now “dig a tunnel” along  $\zeta$  from  $p$  to its “infinite end”. Namely, at each intersection of  $a$  with another curve  $c$  of  $\Gamma_i$ , we split  $c$  into two new curves, and leave a small gap between the two resulting curves, for  $a$  to pass through (see Figure 5).

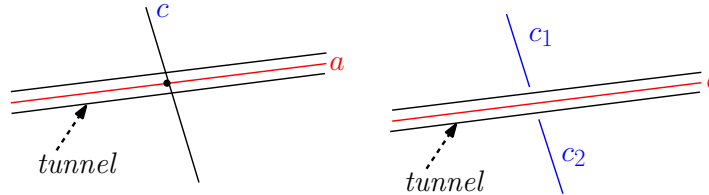


Figure 5: Extending the tunnel. The picture before (left) and after (right).

By construction, during our traversal of  $\zeta$ ,  $a$  did not meet  $F$  again. Thus, as a result of our transformation we have only enlarged  $F$ , increased its complexity, and connected it to an infinite face. Notice that no new intersections are created. Namely, the resulting curves do not self-intersect and if the curves in the original problem intersected pairwise no more than  $s$  times, then none of the newly created curves will intersect pairwise more than  $s$  times. The number of curves in the resulting picture is at most  $1 + (s + 1)(n - 1) = O(sn)$ , if we did not have to cut at  $y$ ; the remaining case is similar.

We are almost done — the trouble is that in case (a), by splitting an existing curve, we cut a bi-infinite curve into semi-infinite curves or even finite sections; similar complications arise in case (b). In case (a), we fix this by extending infinite non-crossing “tails” along  $a$  to infinity in such a way that they follow infinitesimally close to  $a$  but do not cross pairwise. Case (b) is handled analogously.



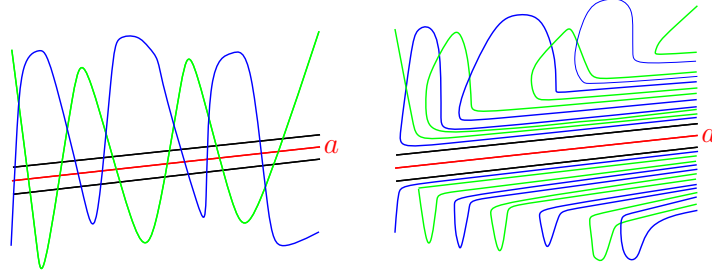


Figure 6: Fixing the tunnel. The picture before (left) and after (right) in the case of  $\Gamma_0$ , magnified.

At each split  $p$ , we would add one infinite “tail” to the finite sections, which were created as a result of the original split. Refer to Figure 6. We note that  $\lambda_s(kn) = O(\lambda_s(n))$  for any constants  $s$  and  $k$ . Thus by Theorem 6, the complexity of  $F$  in cases (a), (b), and (c) is  $O(\lambda_s(n))$ ,  $O(\lambda_{s+1}(n))$ , and  $O(\lambda_{s+2}(n))$ , respectively.  $\square$

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